Flexoelectric switching in a bistable nematic device

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We present a continuum theory model of switching in a bistable nematic liquid crystal device. The bistability of the device investigated relies on the fact that one of the cell surfaces exhibits two stable anchoring states, that is, two surface director orientations are locally stable. Since the other surface exhibits monostable, homeotropic anchoring there are two possible ground state director orientations within the cell, depending on the director orientation at the bistable surface. We first investigate the stability of these base states and find a critical surface anchoring strength below which only one of the states is stable. We also investigate the process of switching between the two stable states through the application of an electric field and the presence of a flexoelectric polarization. At high field strengths the dielectric interaction with the applied field will dominate the flexoelectric effect and may hinder switching. We find, therefore, that a *window* of possible field strengths exists within which switching occurs.

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I. INTRODUCTION

With the dramatic increase in demand for portable electronic devices such as mobile phones, personal organizers, computers, and games consoles, there has recently been a significant amount of research undertaken with the aim to decrease the power consumption and thus increases the battery lifetime, which ultimately affects the portability of such a device. A significant factor in such a consideration is the power consumption of the display, and considerable research effort has been devoted to reducing the operating voltages and minimizing the time over which a voltage must be applied to the liquid crystal display (LCD) commonly used in such devices.

A significant reduction in power consumption is achieved by using a *bistable* display. In a bistable display, as in a monostable LCD, a layer of liquid crystal material is sandwiched between two glass or plastic substrates across which a voltage may be applied, independently to each pixel of the display. This applied electric field may alter the molecular configuration of the layer of liquid crystal and thus alter the optical characteristics of the display and, usually with the aid of certain optical filters, switch between a light and dark state. However, in contrast to a monostable LCD, in a bistable LCD the two molecular configurations corresponding to the light and dark states are locally stable when the applied voltage is removed. Therefore, power is only needed to switch from one stable state to another, in contrast to the monostable LCD, which requires power to switch between and *maintain* the light and dark states. For electronic devices in which the image or parts of the image remain in a fixed state for some time, this would dramatically reduce the number of pixels to be addressed in one refresh cycle and correspondingly reduce the power consumption of the device.

Possibly the most researched bistable LCD technology has been the ferroelectric liquid crystal display [1]. Such a display exhibits two locally stable configuration states in both of which the optic axis is contained within the plane of the cell (and thus leads to favorable optical characteristics) and allows fast switching between states so that video rate addressing is possible. However, the shock stability of such a display can be extremely poor, with the slightest external pressure destroying the delicate smectic layering structure and ruining the displayed image. In recent years alternative bistable LCDs have been investigated in order to combat this problem [2–6]. These alternatives have all been based on the more traditional nematic LCD technologies that have been extremely successful in the display market over the last 25 years. The ability to use existing production methods and liquid crystal materials is an added bonus when considering the cost effectiveness of such new technologies. These nematic devices differ from the twisted and super twisted nematic displays commonly used in that, they usually contain a surface treatment or surface morphology, which leads to bistability. One such bistable LCD technology is the zenithal bistable device (ZBD) [5,6], which makes use of a nonplanar substrate exhibiting a grating morphology and allows two optically distinct director structures. The presence of the complex substrate morphology makes theoretical modeling of the ZBD display extremely difficult, and while some numerical modeling has been carried out to show the configuration of the static bistable states [5] there has been little theoretical research into the mechanisms involved in switching between these states.

In this paper we construct and investigate a model of a bistable nematic device in which one of the substrates exhibits zenithally bistable anchoring. This bistable surface allows both homeotropic and planar anchoring, with both of these surface states being of equal energy and locally stable. When the opposite substrate exhibits fixed homeotropic anchoring there are, therefore, two possible director configurations within the device depending on the orientation of the director at the bistable substrate. In this paper we do not suggest a surface treatment that would produce such bistability but simply investigate the switching characteristics of such a device. We also propose that this model may be a good approximation of the ZBD display described above. Our model does not include the exact surface morphology of the ZBD display but approximates the substrate by a planar surface, which allows two alignment states, as in the ZBD display. If this approach is used as a model of the ZBD, careful consid-

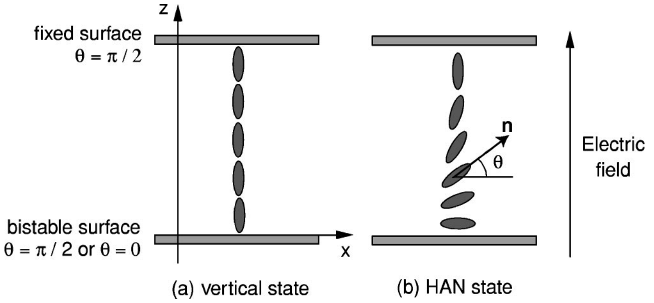


FIG. 1. The simple bistable cell. The lower substrate exhibits bistable anchoring such that both the $\theta=0$ and $\theta=\pi/2$ orientations are stable. With fixed homeotropic anchoring at the upper substrate there exist two possible director configurations within the cell. (a) The vertical state, where $\theta\equiv\pi/2$ throughout the cell, and (b) the hybrid aligned nematic (HAN) state, where the director varies from a homeotropic to planar orientation through the cell.

eration must be given to the exact nature of this *approximate surface*, and the importance of properties such as the surface relaxation coefficient will be discussed later in this paper.

In our theoretical model, the director (the unit vector **n** describing the average local alignment of molecules in the liquid crystal) at one substrate is fixed at an angle of $\theta=\pi/2$ from the horizontal (strong homeotropic anchoring), whereas at the other substrate a bistable surface energy allows two stable states, $\theta=0$ or $\theta=\pi/2$ (as shown in Fig. 1). The two bistable states are, therefore, the *vertical state*, in which the bistable surface of the cell is anchored homeotropically so that the director is in the z direction throughout the cell, and the *hybrid aligned nematic* (HAN) *state*, where the bistable surface of the cell exhibits planar anchoring and the bulk of the cell contains a linear gradient in θ from homeotropic orientation ($\theta=\pi/2$) at the upper surface to planar orientation ($\theta\approx 0$) at the lower surface. In a subsequent section we see that, due to the elastic distortion within the HAN state, the director at the lower surface is not exactly in the planar orientation and, therefore, $\theta(0)$ is nonzero. We will calculate the change in the director orientation $\theta(z,t)$ through the cell as switching between these two states occurs. This switching will be influenced by a variety of factors including flexoelectricity, the voltage across the cell, the length of time for which the voltage is applied (the pulse length), elasticity, viscosity, and anchoring strength.

In most nematic liquid crystal devices it is the dielectric effect that is utilized in switching between the light and dark states. That is, an applied electric field induces a molecular dipole and thus orients the molecules to align with the field direction. How the molecules align with the field is dependent on the difference in dielectric permitivities parallel ($\epsilon_\parallel$) and perpendicular ($\epsilon_\perp$) to the long axis of the molecule. If the difference in permitivities ($\Delta\epsilon=\epsilon_\parallel-\epsilon_\perp$), the dielectric anisotropy, is positive the molecule will align the director **n** parallel to the electric field. If the dielectric anisotropy is negative the induced dipole causes **n** to align perpendicular to the field. This effect is dependent on only the magnitude of the electric field and not the direction, and consequently a positive voltage applied to the cell will have the same effect as a negative voltage.

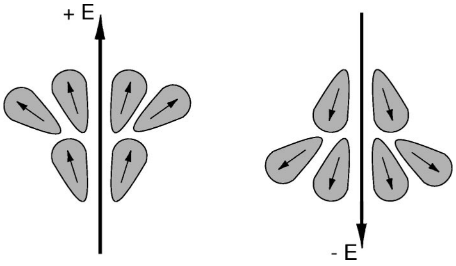


FIG. 2. With pear-shaped asymmetric molecules and a permanent molecular electric dipole an applied field will induce a splay distortion as the molecular dipole aligns along the field lines. An electric field in the opposite direction will induce a different distortion.

From Fig. 1 we see that by applying an electric field in the z direction it may be possible to switch from the vertical state to the HAN state in a negative dielectric material ($\Delta\epsilon<0$) through the dielectric effect since the field will tend to orient the director perpendicular to the field direction. For a positive dielectric material ($\Delta\epsilon>0$), switching from the HAN state to the vertical state may be possible through the dielectric effect since the field tends to orient the director parallel to the field direction. An important question is "How does an applied electric field switch the cell in the opposite way?" That is, from HAN to vertical in a $\Delta\epsilon<0$ material and from vertical to HAN in a $\Delta\epsilon>0$ material. In this paper we investigate the possibility that the *flexoelectric* effect [7,8] may counteract the dielectric effect and enable bistable switching between both states.

In general a nematic liquid crystal molecule is asymmetric, for instance, the pear-shaped molecules depicted in Fig. 2, and contains a small permanent electric dipole. When an electric field is applied to such a liquid crystal material, the small permanent molecular dipoles will align with the field and in doing so induce a splayed distortion due to the molecular shape. For an electric field in the opposite direction the induced distortion is reversed. Another consequence of the molecular asymmetry and permanent dipole is that if a splay distortion is induced, a macroscopic permanent dipole—the sum of the molecular dipoles—will form within the material. A similar effect occurs, with a corresponding *bend* distortion, for bent or banana shaped molecules. An important aspect of this *flexoelectric* effect is that a positive electric field and a negative field induce *different* distortions. We will show that by utilizing the flexoelectric effect we may switch the bistable cell in Fig. 1 between the HAN and vertical states for both positive and negative dielectric materials. Whereas previous theoretical research into the flexoelectric effect has concentrated on the effects on cholesteric helix distortion [9] and monostable liquid crystal cells [10], we will consider a bistable device in which flexoelectricity is of crucial importance when switching between states.

II. THE MODEL

We intend to model the bistable device described in the preceding section using a simplified version of the nematic continuum theory of Ericksen and Leslie [11,12] in which the fluid flow is neglected while the angular motion of the director is retained. In doing so it is necessary to constructing the free energy of the liquid crystal within the cell, which will be a function of the director $\mathbf{n}(z,t)$ at position z through the cell and at time t. The free energy will contain components describing the elastic energy of director distortions within the liquid crystal layer, an approximate electrostatic energy that will include the flexoelectricity of the material, and the surface anchoring energy of the bistable surface.

In constructing the free energy of the cell we will make certain simplifying assumptions in order to reduce the number of model parameters and to ensure that the resulting differential equations are analytically tractable. We use the common ‘‘one-constant approximation’’ for the elastic energy so that the liquid crystal is assumed to be elastically isotropic. Such an approximation may be questionable if the liquid crystal layer contained any twist distortion of the director since the twist elastic constant is usually significantly smaller than the bend and splay elastic constants. However, since we only consider splay and bend distortion of $\mathbf{n}$ we may be confident that such an approximation leads to at least qualitatively correct behavior.

Possibly the most significant approximation used in constructing the free energy is in the form of the electrostatic energy. We will assume that the electric field is constant throughout the cell (and in the direction perpendicular to the cell surfaces), whereas, in reality, it is the displacement field $\mathbf{D}$ that is constant through the cell and the electric field should be calculated by solving Maxwell's equations. We have used such an approximation in order to simplify the governing equations and obtain analytic results for quantities such as the critical switching field strength but we note that such an approximation will be acceptable if the dielectric anisotropy of the liquid crystal is small. With such an approximation, the electric field strength may be related to the total voltage across the cell by $E=V/d$, where d is the thickness of the liquid crystal layer. However, in this paper we will subsequently refer to the magnitude of the electric field E rather than the voltage applied across the cell.

These energy terms form the bulk free energy

$$F_{bulk}=\int_{\mathcal{V}}\left[\frac{K}{2}\{(\boldsymbol{\nabla}\cdot\mathbf{n})^2+(\boldsymbol{\nabla}\times\mathbf{n})^2\}-\frac{\epsilon_0\Delta\epsilon}{2}(\mathbf{E}\cdot\mathbf{n})^2-\mathbf{P}\cdot\mathbf{E}\right]dv, \tag{1}$$

where $\mathcal{V}$ is the region of the liquid crystal within the cell. The first term in Eq. (1) is the elastic energy where K is the isotropic elastic constant. The second term is the dielectric electrostatic energy, where $\epsilon_0=8.854\times10^{-12}\ \mathrm{C^2/N\,m^2}$ is the permittivity of free space, $\Delta\epsilon$ is the dielectric anisotropy of the liquid crystal material, and $\mathbf{E}$ is the electric field across the cell. The third term in Eq. (1) is the flexoelectric energy, where $\mathbf{P}=e_{11}(\boldsymbol{\nabla}\cdot\mathbf{n})\mathbf{n}+e_{33}(\boldsymbol{\nabla}\times\mathbf{n})\times\mathbf{n}$ is the flexoelectric polarization [8] present due to the permanent molecular dipole and molecular asymmetry described in the preceding section.

The total free energy also includes an energy term describing the preference for the bistable surface to exhibit either of the two stable director configurations. Such a surface energy term must exhibit energy minima at two distinct director configurations, the planar and homeotropic states. The simplest form of such an energy term is similar to the commonly used Rapini-Papoular surface energy [13], that is, a sinusoidal function of the director orientational angle such that the minima occur at the stable director orientations.

If the director remains within the plane of the two bistable director configurations, we define θ as the angle between the director and the plane of the cell surfaces so that the director is $\mathbf{n}=(\cos\theta,0,\sin\theta)$ as in Fig. 1. Assuming that the director is homogeneous in the plane of the cell and depends only on the coordinate along the cell surface normal we may, therefore, assume that the director angle is a function of only the distance through the cell and time, $\theta=\theta(z,t)$.

With these assumptions the total free energy (per unit area in the xy plane), the sum of the bulk and surface energy terms, is

$$F=\int_0^d\frac{1}{2}\left[K\left(\frac{\partial\theta}{\partial z}\right)^2-\epsilon_0\Delta\epsilon E^2\sin^2\theta-(e_{11}+e_{33})\times E\frac{\partial\theta}{\partial z}\sin(2\theta)\right]dz+W_0\sin^2[2\theta(0,t)], \tag{2}$$

where d is the thickness of the liquid crystal layer in the cell.

As previously mentioned, the surface energy term in Eq. (2) exhibits minima at $\theta(0,t)=0$ or $\theta(0,t)=\pi/2$, that is, the director at $z=0$ is planar or homeotropically anchored. The parameter W_0 is known as the anchoring strength and is a measure of the depth of these energy minima. At this point we note that the flexoelectric term in Eq. (2) is such that if $\theta>0$ a positive electric field can reduce the free energy by inducing a positive gradient in θ and a negative field by inducing a negative gradient in θ. A useful simplification of the flexoelectric energy in this system can be seen by directly integrating the flexoelectric term in Eq. (2) to obtain the following free energy, up to an additive constant that will not appear in the subsequent minimization of the energy:

$$F=\int_0^d\frac{1}{2}\left[K\left(\frac{\partial\theta}{\partial z}\right)^2-\epsilon_0\Delta\epsilon E^2\sin^2\theta\right]dz-\frac{(e_{11}+e_{33})}{4}E\cos[2\theta(0,t)]+W_0\sin^2[2\theta(0,t)]. \tag{3}$$

Thus, the presence of flexoelectricity has effectively altered the surface energy. The original surface energy $W_0\sin^2[2\theta(0,t)]$ has minima at the planar, $\theta(0,t)=0$, and homeotropic, $\theta(0,t)=\pi/2$, surface states (see Fig. 3). However, the flexoelectric term $[-(e_{11}+e_{33})/4]E\cos[2\theta(0,t)]$ has a minimum at $\theta(0,t)=0$ and a maximum at $\theta(0,t)=\pi/2$ when $(e_{11}+e_{33})E>0$ and, conversely, a maximum at $\theta(0,t)=0$ and a minimum at $\theta(0,t)=\pi/2$ when $(e_{11}+e_{33})E<0$. Thus, when $(e_{11}+e_{33})E>0$ the flexoelectric effect destabilizes the homeotropic surface state and stabilizes the planar surface state, but when $(e_{11}+e_{33})E<0$ the opposite occurs.

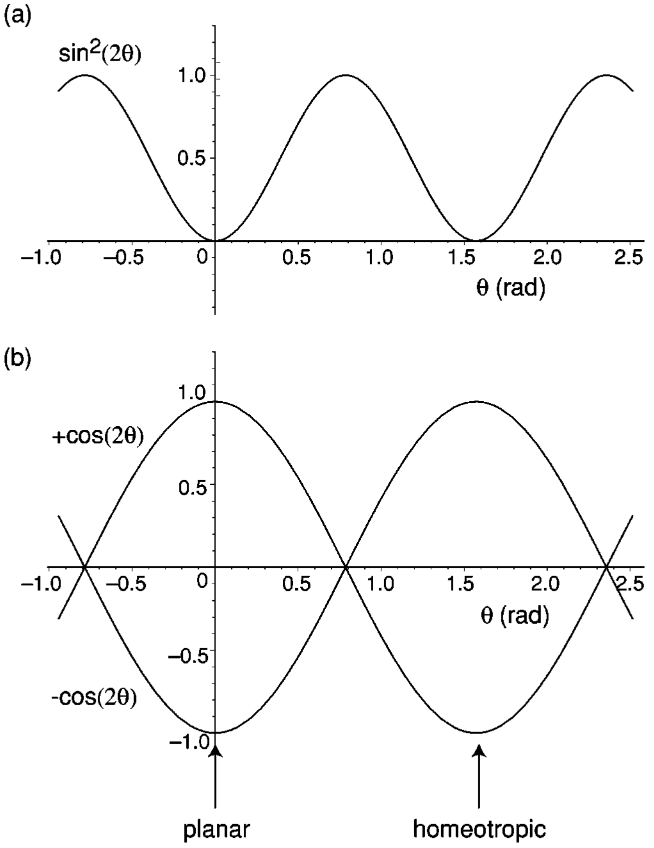


FIG. 3. A graphical representation of the effective surface energy. (a) The bistable anchoring leads to a term with minima at $\theta=0$ and $\theta=\pi/2$. (b) The flexoelectric effect leads to a term that may destabilize the homeotropic $\theta=\pi/2$ state for $(e_{11}+e_{33})E>0$ or destabilize the planar $\theta=0$ state for $(e_{11}+e_{33})E<0$.

Considering the sum of these two surface energy terms we see that, if we initially assume that $e_{11}+e_{33}>0$, then the $\theta(0,t)=0$ state is stable when $E>-8W_0/(e_{11}+e_{33})$ and the $\theta(0,t)=\pi/2$ state is stable for $E<8W_0/(e_{11}+e_{33})$. For $e_{11}+e_{33}<0$ the same effect occurs for the opposite signs of the electric field. This stabilization/destabilization of the surface states will be clearly seen in the subsequent analytic and numerical results. Thus, through the flexoelectric effect, we can control the stability of the surface orientation states by altering the magnitude and direction of the applied electric field.

Using this free energy equation, (3), and neglecting the fluid velocity in the Ericksen-Leslie equations [11,12] we obtain the governing equation for the director motion in the bulk of the cell,

$$\mu\frac{\partial\theta}{\partial t}=K\frac{\partial^2\theta}{\partial z^2}+\frac{\epsilon_0\Delta\epsilon}{2}E^2\sin(2\theta), \tag{4}$$

where μ is the viscosity associated with rotation of the director. From variational principles, at the bistable surface of the cell $(z=0)$ we obtain the boundary condition

$$\mu_s\frac{\partial\theta}{\partial t}(0,t)=K\frac{\partial\theta}{\partial z}(0,t)-2W_0\sin[4\theta(0,t)]-E_{13}E\sin[2\theta(0,t)], \tag{5}$$

where we have defined an average flexoelectric coefficient $E_{13}=(e_{11}+e_{33})/2$ and μ_s is the surface relaxation coefficient. The surface relaxation coefficient is sometimes called the surface viscosity and may be dimensionally compared to the bulk viscosity by $\mu_s=\gamma\mu$, where γ_s is a "surface length" thought to be of the order of the molecular length. Therefore, the surface relaxation coefficient is usually considered to be significantly smaller then the bulk viscosity. However, as we discuss in Sec. VI, in the model of the ZBD display the effective surface relaxation coefficient may be considerably larger. At the other cell surface $(z=d)$ we will assume infinite anchoring in the homeotropic state so that

$$\theta(d,t)=\frac{\pi}{2}. \tag{6}$$

The following sections will be devoted to solving Eq. (4) subject to the boundary conditions in Eqs. (5) and (6). However, we can make some qualitative observations as to what we expect to happen in such a system.

The first point to note is that the dielectric interaction between the electric field and the director is proportional to E^2 while the flexoelectric interaction depends linearly on E. Therefore, we may expect the flexoelectric interaction to dominate at low fields and the dielectric interaction to dominate at high field strengths.

With a positive dielectric anisotropy $(\Delta\epsilon>0)$ the dielectric interaction will tend to align the director with the electric field in the z direction. Therefore, the vertical state will be favored by the dielectric interaction and switching from the HAN state to the vertical state should be relatively straightforward. As long as the field strength is sufficiently high to break the planar surface anchoring and dominate any flexoelectric effect it should be possible to switch to the vertical state with a positive or negative field strength (switching is independent of the polarity of the electric field since the dielectric interaction depends on E^2). We have also noted that for $E>0$ the flexoelectric term stabilizes the planar surface state while the homeotropic surface state is destabilized, and for $E<0$ the flexoelectric term stabilizes the homeotropic surface state and destabilizes the planar surface state. Therefore, in order to switch from the vertical state to the HAN state we will need to apply a positive field, to destabilize the homeotropic anchoring and stabilize the planar anchored state, of a strength that is sufficiently high to break the homeotropic anchoring but sufficiently low so that the flexoelectric effect dominates the dielectric effect.

With a negative dielectric anisotropic material, $\Delta\epsilon<0$, a similar effect occurs. In this case the dielectric effect tends to align the director perpendicular to the electric field. Thus switching from the vertical state to the HAN state should be possible through the dielectric effect with a sufficiently high positive or negative field. In a similar way to the $\Delta\epsilon>0$ case switching from the HAN state to the vertical will require an intermediate negative electric field strength.

One other point is worth mentioning at this stage. There exists a parameter symmetry with the above equations, which means we need only consider positive flexoelectric coefficients, $E_{13}>0$. We notice that if E_{13} is replaced by $-E_{13}$ and E is replaced by $-E$ the equations remain unchanged. This symmetry will be evident in the following sections.

III. STABILITY OF BASE STATES

In order for the device discussed in this paper to be used as a bistable nematic display it is clearly necessary for the cell to exhibit two locally stable static director configuration states at zero field, the vertical and HAN states. The vertical state contains an undistorted director configuration, $\theta(z)\equiv\pi/2$, while in the HAN state there exists some distortion as the director rotates from the homeotropic state at the upper surface to the planar state at the lower surface, see Fig. 1. In fact the director will not exhibit an exactly planar configuration at the lower surface since there exists a torque on the surface director due to the elastic distortion in the bulk of the cell. To minimize the elastic free energy the director would prefer to be in an undistorted configuration and thus the director at the lower surface feels a torque tending to align it with the fixed homeotropic director at the upper surface. If the lower surface anchoring strength is strong enough, the surface anchoring torque is sufficient to balance the elastic torque and the HAN state is maintained. However, if the surface anchoring is too weak, the elastic torque will break the planar anchoring and the cell will spontaneously switch to the vertical state.

We can see this effect by solving the time-independent version of Eq. (4) with no applied electric field, $E=0$, and subject to the boundary conditions in Eqs. (5) and (6). The solution of this relatively simple equation is $\theta(z)=\pi/2+[\theta(0)-\pi/2](1-z/d)$, where $\theta(0)$ is determined by the equation

$$0=\theta(0)-\frac{\pi}{2}+\frac{2W_0d}{K}\sin[4\theta(0)]. \tag{7}$$

One solution of Eq. (7) is clearly $\theta(0)=\pi/2$, the vertical configuration. If we rearrange Eq. (7) we find

$$\frac{W_0d}{K}=\frac{\pi-2\theta(0)}{4\sin[4\theta(0)]}, \tag{8}$$

which is plotted in Fig. 4. We see that there exists another solution such that $\theta(0)\approx 0$, the HAN state, if W_0 is greater than a critical value. The third solution where $\theta(0)$ takes a value between the planar and homeotropic state leads to an unstable director configuration and will not be considered further. Figure 4 also indicates that as the anchoring strength W_0 tends to infinity, the director at the lower surface in the HAN state tends to the planar configuration. Solving numerically for the critical point it is found that the anchoring strength must satisfy $W_0>0.5758K/d$ to maintain a HAN state. An anchoring strength less than this critical value will result in the vertical state being the only stable state and the cell would not achieve bistability.

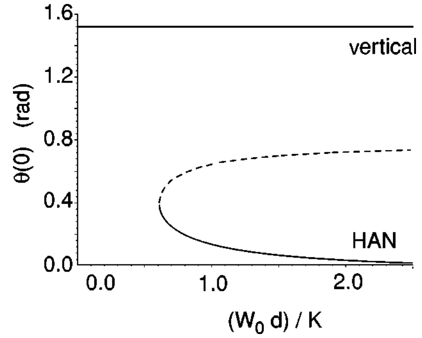


FIG. 4. Stability of the HAN and vertical states at zero electric field. For an anchoring strength $W_0>0.5758K/d$ the HAN state, where $\theta(0)$ is close to zero, is achievable. Solid line denotes stable solutions and dashed line indicates an unstable solution.

IV. STATIC SOLUTIONS

The partial differential equation, (4), for the dynamics of switching in the bistable cell will later be solved numerically, however the static equilibrium configurations may be found analytically by solving the associated time-independent problem for $\theta(z)$,

$$0=K\frac{d^2\theta}{dz^2}+\frac{\epsilon_0\Delta\epsilon}{2}E^2\sin(2\theta), \tag{9}$$

where at the bistable surface of the cell $(z=0)$, we have the boundary condition

$$0=K\frac{d\theta}{dz}(0)-2W_0\sin[4\theta(0)]-E_{13}E\sin[2\theta(0)] \tag{10}$$

and at the other cell surface $(z=d)$

$$\theta(d)=\frac{\pi}{2}. \tag{11}$$

The bulk equation (9) is the static sine-Gordon equation and is known to exhibit solutions involving elliptic functions [14]. The difficulty in the present situation is the complicated boundary condition at $z=0$, Eq. (10). However, we may still write down the solution of this equation in terms of an integration constant that must be found numerically. Solutions to Eq. (9) are

$$\theta_\pm(z)=\frac{\pi}{2}-\sin^{-1}\left[\mathrm{sn}\left(\pm\sqrt{\frac{(-\epsilon_0\Delta\epsilon)E^2}{K\kappa^2}}(d-z),\kappa\right)\right], \tag{12}$$

where $\mathrm{sn}(u,\kappa)$ is an elliptic function defined as the inverse of the elliptic integral of the first kind $F(u,\kappa)$ [14] and κ is the constant function of $\theta(0)$ given by

$$\kappa^2=\frac{-\epsilon_0\Delta\epsilon KE^2}{\{2W_0\sin[4\theta(0)]+E_{13}E\sin[2\theta(0)]\}^2-\epsilon_0\Delta\epsilon KE^2\sin^2[\theta(0)]}. \tag{13}$$

The value of $\theta(0)$ is found by numerically solving the equation

$$\theta(0)=\frac{\pi}{2}-\sin^{-1}\left[\mathrm{sn}\left(\pm\sqrt{\frac{(-\epsilon_0\Delta\epsilon)E^2}{K\kappa^2}}d,\kappa\right)\right]. \tag{14}$$

There are a couple of points to note in this solution. From Eq. (13) we see that if $\Delta\epsilon<0$ then $\kappa^2>0$ and thus κ is a real number so that the argument and modulus of the elliptic function in the solution Eq. (12) are both real. However, if $\Delta\epsilon>0$ it is possible that $\kappa^2<0$ so that κ is an imaginary number and thus the modulus of the elliptic function in Eq. (12) is imaginary. However, this does not imply an imaginary solution since the properties of elliptic functions [14] give

$$\mathrm{sn}(u,ik)=\frac{1}{\sqrt{1+k^2}}\mathrm{sd}\left(u\sqrt{1+k^2},\frac{k}{\sqrt{1+k^2}}\right), \tag{15}$$

where $\mathrm{sd}(u,k)$ is the elliptic function defined in terms of standard elliptic functions as

$$\mathrm{sd}(u,k)=\frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}. \tag{16}$$

The static solution in Eq. (12) also indicates that there are two possible solutions associated with the positive or negative argument of the elliptic function (which we have denoted by $\theta_\pm$). These two solutions, θ_+ and θ_-, correspond to two functions for which $d\theta/dz<0$ and $d\theta/dz>0$, respectively. While for certain parameter values one of these solutions may satisfy both boundary condition at $z=0$ and $z=d$, for other parameter values (in our situation we will see that this occurs when $\Delta\epsilon<0$) one of the solutions satisfies the boundary condition at $z=0$ and the other satisfies the boundary condition at $z=d$. In this case a composite solution must be constructed so that

$$\theta=\begin{cases}\theta_+ & \text{for}\quad 0\leqslant z\leqslant z^*\\ \theta_- & \text{for}\quad z^*<z\leqslant d,\end{cases} \tag{17}$$

where $z=z^*$ is the point where $d\theta_+/dz=0=d\theta_-/dz$.

Solutions for $\Delta\epsilon<0$ and $\Delta\epsilon>0$ are shown in Figs. 5 and 6, respectively, for the parameter values $K=1.63\times10^{-11}$ N, $\Delta\epsilon=\pm5.7$, $d=1$ μm, $E_{13}=4\times10^{-9}$ C/m, $W_0=1.63\times10^{-5}$ N/m. As discussed in the preceding section, the case for negative flexoelectric coefficient, $E_{13}=-4\times10^{-9}$ C/m, would give the same static solutions for the opposite field strengths.

For $\Delta\epsilon<0$, Fig. 5 shows the director configuration for both the stable vertical and HAN states when $E=0$ as well as the director configurations for various positive and negative electric field strengths. For negative field strengths we see that, as predicted in the Sec. II, the $\theta(0)=\pi/2$ surface state is stabilized while the dielectric interaction in the bulk of the cell tends to align the director perpendicular to the field, that is, in the x direction so that $\theta\approx0$. For positive field strengths the $\theta(0)=0$ surface state is stabilized and again the bulk of the cell is aligned so that $\theta\approx0$.

From Fig. 5 we would imagine that, for a positive field strength, if the static electric field is removed, the cell would relax to the HAN state. However, it is not as straightforward for the negative field strength case. The director at $z=0$ would tend to relax the director configuration into the vertical state but the bulk of the cell is close to $\theta=0$ and may force the system to relax to the HAN state. Whether the

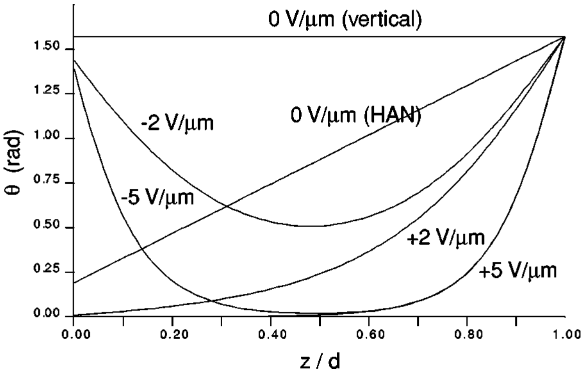


FIG. 5. Static director configuration for $\Delta\epsilon<0$. When $E=0$ the HAN and vertical states are stable. For nonzero field strengths the dielectric effect in the bulk of the cell causes a planar director alignment, $\theta\approx0$. The flexoelectric effect stabilizes the planar surface alignment $\theta(0)=0$ for $E>0$ and the homeotropic surface alignment $\theta(0)=\pi/2$ for $E<0$.

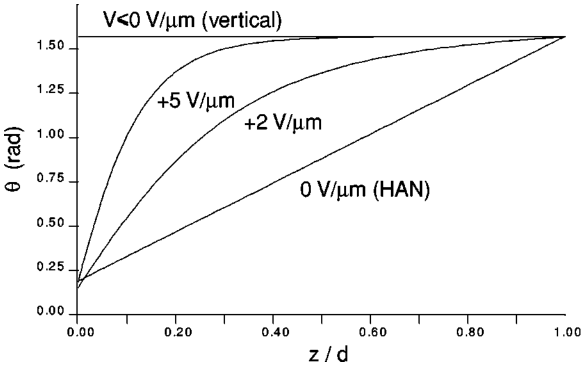


FIG. 6. Static director configuration for $\Delta\epsilon>0$. When $E=0$ the HAN and vertical states are stable. For nonzero field strengths the dielectric effect in the bulk of the cell causes a homeotropic director alignment, $\theta\approx\pi/2$. The flexoelectric effect stabilizes the planar surface alignment $\theta(0)=0$ for $E>0$ and the homeotropic surface alignment $\theta(0)=\pi/2$ for $E<0$. Therefore, for all $E<0$ the director configuration is exactly the vertical state $\theta(z)\equiv0$.

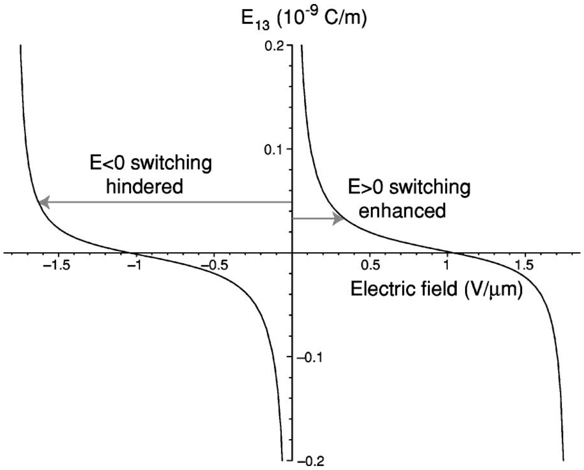


FIG. 7. Analytic results for the critical field strength above which distortion of the vertical state occurs when $\Delta\epsilon<0$. For $E_{13}>0$ the flexoelectric effect reduces the critical field strength when $E>0$ and increases the magnitude of the critical field strength when $E<0$. The reverse effect occurs when $E_{13}<0$.

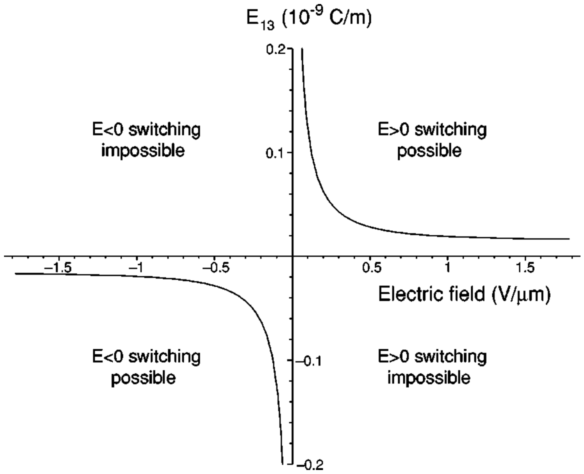


FIG. 8. Analytic results for the critical field strength above which distortion of the vertical state occurs when $\Delta\epsilon>0$. For $E_{13}>0$ the flexoelectric effect enables switching to occur when $E>0$ while for $E_{13}<0$ switching only occurs for $E<0$.

surface anchoring alignment or the elastic realignment in the bulk of the cell dominates after the field is removed will be further investigated when we numerically solve the dynamic equations.

For $\Delta\epsilon>0$, Fig. 6 shows the static equilibrium director configurations for various field strengths. In this case the dielectric interaction in the bulk of the cell causes the director to align with the electric field, i.e., such that $\theta\approx\pi/2$. For a positive field strength the flexoelectric interaction stabilizes the $\theta(0)=0$ surface state while, for a negative field strength the flexoelectric interaction stabilizes the $\theta(0)=\pi/2$ surface state. Thus, for a positive field strength there is a competition between the bulk and surface orientation, as in the $\Delta\epsilon<0$ case, and the cell may relax to either the HAN or vertical state. In contrast, the solution for all negative field strengths is simply the vertical state.

In summary, for $\Delta\epsilon<0$ a positive field strength will always lead to the HAN state while a negative field strength may lead to either the HAN state or the vertical state. For $\Delta\epsilon>0$ a negative field strength will always lead to the vertical state while a positive field strength may lead to either the vertical state or the HAN state.

V. VERTICAL TO HAN SWITCHING

Before resorting to a numerical solution of Eq. (4) we can make some analytical progress by considering the switching from the vertical to HAN state. Since the vertical state has a simple mathematical form $\theta(z)\equiv\pi/2$ and switching to the HAN state occurs at a threshold similar to the classical Freedericksz transition it is possible to analytically calculate the critical field strength for this transition using a perturbation expansion. To investigate the HAN to vertical switching in such a way would be impossible since there is no distinct threshold field so that the HAN director configuration will distort when any nonzero field strength is applied.

For vertical to HAN switching we proceed using the static versions of the bulk equation (4) and the boundary conditions, Eqs. (5) and (6), and in looking for the critical field strength we assume that at a field strength slightly above the critical value $E=E_c+O(\delta)$, for a small parameter $\delta\ll 1$, the director configuration will be perturbed from the vertical state so that

$$\theta=\frac{\pi}{2}-\delta\theta_1(z). \tag{18}$$

Thus $\delta\theta_1(z)$ is the perturbation from the vertical state, and upon linearization Eq. (4) gives

$$0=K\frac{d^2\theta_1}{dz^2}-\epsilon_0\Delta\epsilon E_c^2\theta_1, \tag{19}$$

the solution of which is easily found in terms of exponential or sinusoidal functions depending on whether $\Delta\epsilon>0$ or $\Delta\epsilon<0$, respectively.

The boundary conditions (5) and (6) then lead to the condition that

$$E_{13}=\frac{4W_0}{E_c}+\frac{\sqrt{-\epsilon_0\Delta\epsilon K}}{2\tan\left(\sqrt{\frac{-\epsilon_0\Delta\epsilon}{K}}E_c d\right)} \tag{20}$$

if $\Delta\epsilon<0$ or

$$E_{13}=\frac{4W_0}{E_c}+\frac{\sqrt{\epsilon_0\Delta\epsilon K}}{2\tanh\left(\sqrt{\frac{\epsilon_0\Delta\epsilon}{K}}E_c d\right)} \tag{21}$$

if $\Delta\epsilon>0$.

This gives us a relationship between the critical field and the flexoelectricity as shown in Figs. 7 and 8 for the same parameter values used in the preceding section: $K=1.63\times10^{-11}$ N, $\Delta\epsilon=\pm5.7$, $d=1$ μm, $W_0=1.63\times10^{-5}$ N/m.

For the case of a negative dielectric anisotropy (Fig. 7) with no flexoelectricity ($E_{13}=0$), the critical field for positive E equals the critical field for negative E and introducing flexoelectricity breaks this symmetry so that for $E_{13}<0$, switching with a positive field occurs at a *higher* field strength than with a negative field strength. For $E_{13}>0$ switching with a positive field occurs at a *lower* field strength than with a negative field strength. This behavior agrees well with the qualitative argument discussed in Sec. II. For $\Delta\epsilon<0$ the dielectric effect alone will be able to switch from the vertical to HAN state. However, the effect of flexoelectricity in this situation is to stabilize the vertical state (for $E_{13}E<0$) and thus hinder switching from the vertical state, or destabilize the vertical state (for $E_{13}E>0$) and thus promote switching.

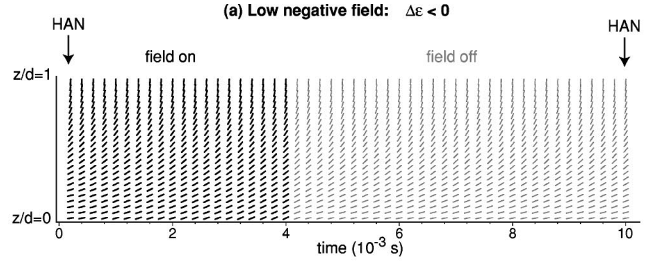
For a positive dielectric anisotropy (Fig. 8), unlike the case with negative dielectric anisotropy, only a positive field strength will switch the cell if $E_{13}>0$ and only a negative field strength will switch the cell if $E_{13}<0$. With $E_{13}>0$ and a negative field strength, both the dielectric effect and the flexoelectric effect prefer, and thus the cell remains in, the vertical state. With positive field strength the dielectric effect still favors the vertical state but the flexoelectric effect favors the HAN state. For negative E_{13} a similar argument applies due to the parameter symmetry discussed in Sec. II.

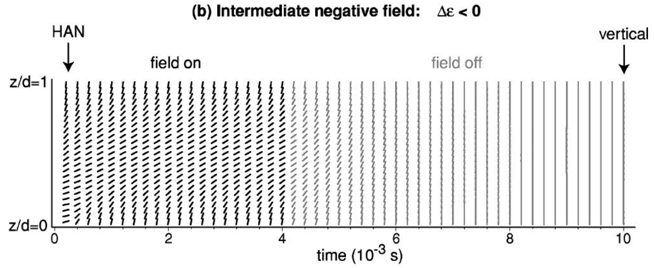
Such a critical field strength is the value below which switching will not occur. As we will see in the following section there are a variety of factors such as elastic and dynamic effects as well as the exact form of the electric field applied (i.e., triangular wave form or pulsed field), which determine whether the cell will switch to another state above the critical field strength determined in this section. The critical field strength we have found is the threshold at which a director distortion occurs. For the cell to switch from the vertical to HAN state the director at the bistable surface must pass the $\theta(0)=\pi/4$ position so that the transition from homeotropic to planar anchoring occurs. To consider the full switching of the cell we must solve the governing equations numerically.

VI. PULSE SWITCHING

In preceding sections, through an understanding of the competition between energy terms and analytical solutions, we have made qualitative predictions as to the behavior of the bistable nematic cell. We have proposed that, for a material such that $\Delta\epsilon<0$, switching from the vertical to HAN state should be achievable with $E>0$ through the dielectric interaction, but may be hindered by the flexoelectric interaction for $E<0$. Switching from the HAN to vertical state will be hard to achieve with $E>0$ since both the dielectric and flexoelectric interactions prefer to remain in the HAN state. It may, however, be possible to switch from the HAN to vertical state with $E<0$ since the flexoelectric effect will favor switching.

For a positive dielectric material, $\Delta\epsilon>0$, the opposite effect is expected. Switching from the HAN to vertical state is expected to be achievable with $E<0$ and less likely with $E>0$. Vertical to HAN switching will be hard to achieve with $E<0$ but may be possible with $E>0$.


(a) Low negative field: $\Delta\varepsilon < 0$


(b) Intermediate negative field: $\Delta\varepsilon < 0$

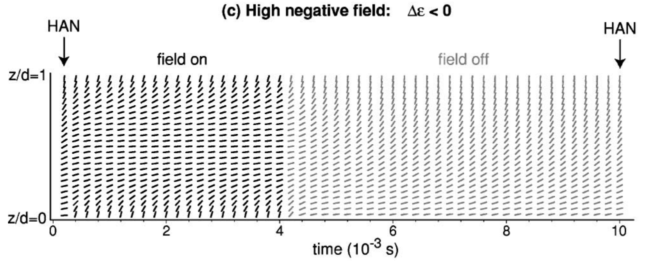

(c) High negative field: $\Delta\varepsilon < 0$

FIG. 9. The director configuration through the cell when a pulsed electric field is applied for three different field strength values. (a) $E=-0.2$ V/μm, the flexoelectric effect is not strong enough to destabilize the planar surface anchoring at $z=0$ and the cell does not switch out of the HAN state. (b) $E=-1$ V/μm, the flexoelectric effect is strong enough to destabilize the planar surface anchoring at $z=0$ and the cell switches from the HAN state to the vertical state. (c) $E=-3$ V/μm, although the flexoelectric effect is sufficient to destabilize the planar surface orientation the dielectric effect within the bulk of the cell creates a torque on the surface director when the field is removed, and the cell returns to the HAN state.

A full investigation of the switching characteristics may only be attempted by solving the full nonlinear partial differential equations (4), (5), and (6). In this section we solve these equations using a standard numerical relaxation method. Initially (at $t=0$) we set the director configuration in either of the two zero-field, static equilibrium states, that is, in either the HAN or the vertical state shown in Figs. 5 and 6. In the model we then apply a single electric field pulse for a fixed time $0<t<t_{pulse}$ during which the electric field is taken to be a constant value $E=E_{pulse}$. The pulse time t_{pulse} is taken to be sufficiently long to allow the director configuration to reach an equilibrium state, the static configuration found analytically in Sec. IV, and then the electric field is removed so that $E=0$. The system is allowed to equilibriate once again and the final state, either HAN or vertical, is recorded. Three such runs of the numerical code are illustrated in Fig. 9.

In order to investigate switching from the HAN to vertical state for a material such that $\Delta\epsilon<0$ we have calculated the

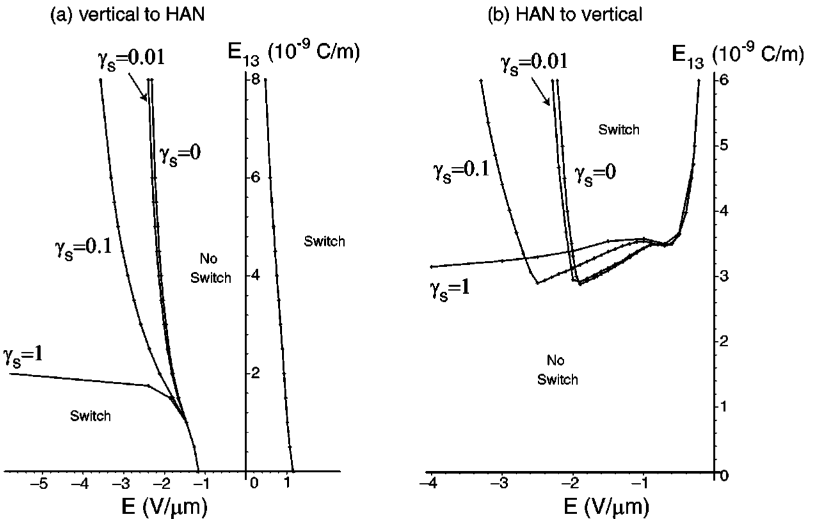


FIG. 10. Switching regions for different flexoelectric coefficients E_{13} and surface relaxation coefficients $\mu_s = \gamma_s \mu$ when $\Delta\epsilon < 0$ and for both vertical to HAN and HAN to vertical switching.

director configuration through the cell, which is initially in the HAN state, as a field pulse is applied and then removed. Figure 9 shows the change in the director configuration as a function of time for three field strengths and the parameter values $K = 1.63\times10^{-11}$ N, $\Delta\epsilon = \pm 5.7$, $d = 1$ μm, $E_{13} = 4\times10^{-9}$ C/m, $W_0 = 1.63\times10^{-5}$ N/m, $\mu = 0.163$ N s/m^2, $\mu_s = 1.63\times10^{-4}$ N s/m. According to the preceding sections such a switch between states is favored by the flexoelectric interaction but not by the dielectric interaction. In Fig. 9(a) a relatively low field, of strength $E = -0.2$ V/μm, is applied and the flexoelectric effect is insufficient to break the planar surface anchoring at $z = 0$ and thus switching does not occur. In Fig. 9(b) the applied field strength ($E = -1$ V/μm) was sufficient for the flexoelectric effect to break the planar surface anchoring and align the director near $z = 0$ in the homeotropic state. Thus, when the field is removed, the system equilibriates to the vertical state and the cell has been switched. In Fig. 9(c) a larger field strength of $E = -3$ V/μm is applied so that, while the surface anchoring is broken and the director close to $z = 0$ is in the homeotropic state, the bulk of the cell is forced, through the dielectric interaction, to lie in a planar orientation. After the field is removed at $t = t_{pulse}$ an elastic torque between the bulk region and the surface region forces the surface director back to the planar state. The cell is forced *back* to the HAN state and the cell does *not* undergo switching. Therefore, while we can achieve switching between the HAN and vertical states for $\Delta\epsilon < 0$, it is only within a fixed range of field strengths. The extent of this switching window will clearly depend on the choice of system parameters and we now investigate the effects of altering two of the more important parameters, the flexoelectric coefficient E_{13} and the surface relaxation coefficient μ_s. Figures 10 and 11 indicate how the regions of switching change with varying E_{13} and μ_s for $\Delta\epsilon < 0$ and $\Delta\epsilon > 0$, respectively, for both cases of switching, from HAN to vertical and vertical to HAN.

If we first consider the switching regions for zero surface relaxation coefficient we see that as indicated at the start of this section, switching from vertical to HAN state is possible in the $\Delta\epsilon < 0$ case [Fig. 10(a)] for $E > 0$ and $E < 0$. As suggested in preceding sections, we see that switching from HAN to vertical state in the $\Delta\epsilon < 0$ case [Fig. 10(b)] is not possible for $E > 0$ while there is a region of switching for $E < 0$ (as we found in Fig. 9 when we fixed $E_{13} = 4$

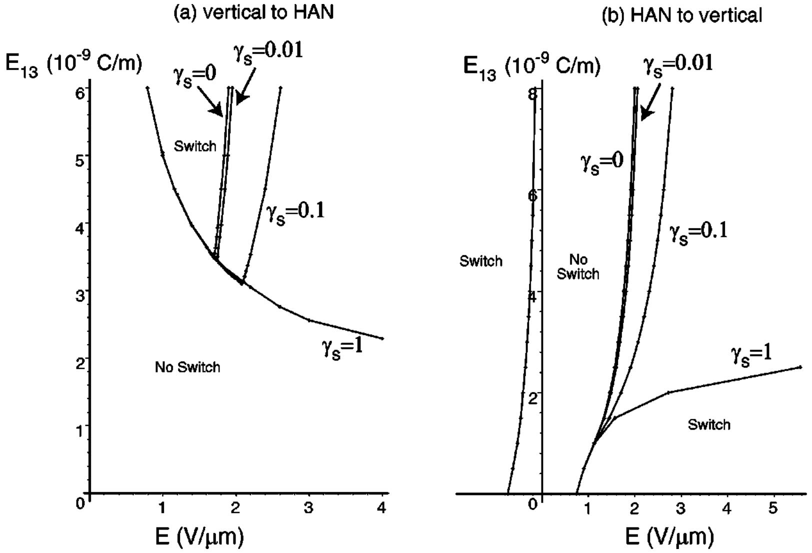


FIG. 11. Switching regions for different flexoelectric coefficients E_{13} and surface relaxation coefficients $\mu_s = \gamma_s \mu$ when $\Delta\epsilon > 0$ and for both vertical to HAN and HAN to vertical switching.

$\times 10^{-9}$ C/m). For larger values of E_{13} the range of field strengths over which switching can occur increases since the flexoelectric interaction may then dominate the dielectric interaction at larger field strengths.

Figure 11 shows the equivalent switching regions for $\Delta\epsilon>0$. For $\mu_s=0$ the range of field strengths over which the cell switches also increases as the flexoelectric coefficient increases. It is interesting to note that in this case the region of switching from the vertical state to the HAN state for $E>0$ [Fig. 11(a)] is smaller than the region of switching from the HAN to vertical state for $E<0$ in the $\Delta\epsilon<0$ case [Fig. 10(b)]. This is because the cell has a fixed homeotropic state at the upper surface $\theta(d)=\pi/2$. The elastic energy within the cell makes it energetically preferable to remain in the vertical state, with zero elastic energy, rather than in the HAN state, which has a nonzero elastic energy. Therefore switching into the vertical state is energetically favored over switching into the HAN state and hence the smaller region of switching.

With a nonzero surface relaxation coefficient, $\mu_s\sim 0.01\mu$, that is, μ_s is very small compared to the bulk viscosity μ, there is very little difference in the switching regions as shown in Figs. 10 and 11. However, as mentioned in Sec. I, if the bistable device considered in this paper is thought of as an approximate model of the zenithal bistable device, the surface is effectively modeling a region of the real device, which includes the grating surface morphology and a region of liquid crystal material. Associated with this simplified, effective surface will be an effective surface relaxation coefficient that may be significantly closer to the bulk viscosity. When the surface relaxation coefficient is closer in value to, or indeed the same as, the bulk viscosity there are significant changes in the switching regions in Figs. 10 and 11.

With a large surface relaxation coefficient the reorientation of the surface director seen in Fig. 9(c) after the field is removed will be a slower process. Thus the director within the bulk has time to reorient, and instead of a final HAN state in Fig. 9(c) the vertical state is obtained. Thus, switching to the vertical state in this situation may be preferred when the surface relaxation coefficient is included. This is seen in Fig. 10(b) where the switching region increases as the surface relaxation coefficient is increased. Such an explanation can be given to explain all the changes in switching regions as the surface relaxation coefficient is increased in Figs. 10 and 11.

When the surface orientation and bulk orientation are the same when a field is applied, i.e., for $E>0$ when $\Delta\epsilon<0$ and for $E<0$ when $\Delta\epsilon>0$, there is no torque between the bulk and surface directors and no surface director motion so that altering the surface relaxation coefficient does not effect the critical switching field strength.

VII. CONCLUSION

In this paper we have examined the effects of flexoelectricity on switching in a one-dimensional, model bistable cell. We have seen that for the cell to be bistable it must have a sufficiently high anchoring strength to maintain stability, and the strength of the field required to switch a cell from one state to another critically depends on the sign and magnitude of the electric field, the magnitude of the flexoelectric coefficient, and the surface relaxation coefficient.

For $\Delta\epsilon<0$ both positive and negative field strength can cause the cell to switch from the vertical state to the HAN state whereas only a specific range of negative field strength will allow the cell to switch from the HAN state to the vertical state. For $\Delta\epsilon>0$ both positive and negative field strength can cause the cell to switch from the HAN state to the vertical state but only a specific range of positive field strength will allow the cell to switch from a vertical state to a HAN state.

Diagrams such as Figs. 10 and 11 lead to a good qualitative understanding of the switching of such a bistable device and it is hoped that in the future we may obtain qualitative and quantitative agreement between our model and experimental results.

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